

# Relation between Magnitude Series Correlations and Multifractal Spectrum Width

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(February 1, 2008)

We study the correlation properties of long-range correlated time series,  $x_i$ , with tunable correlation exponent and built-in multifractal properties. For the cases we investigate we find that the correlation exponent of the magnitude series,  $|x_{i+1} - x_i|$ , is a monotonically increasing function of the multifractal spectrum width of the original series.

PACS numbers: 87.10.+e, 87.80.+s, 87.90+y

Many natural systems exhibit complex dynamics characterized by long-range power-law 2-point correlations [1–3]. In some cases the output of such systems — a series of a fluctuating variable  $x_i$  — may be characterized by a single 2-point correlation exponent, while in other cases the series may be quantified by a spectrum of exponents called the multifractal (MF) spectrum [4–6]. The MF spectrum quantifies in details the long-range correlation properties of a series  $x_i$ . Nonetheless, additional way to measure the correlation properties of a complex series,  $x_i$ , is to analyze the long-range power-law correlations for the magnitudes of the increments  $|\Delta x_i| \equiv |x_{i+1} - x_i|$ .

For example, complex fluctuations in financial indices are characterized by a broad MF spectrum [7]. Moreover, recent reports showed that there are positive long-range power-law correlations in the volatility (magnitude of the increments) in such financial time series [8]. Biomedical time series are also found to exhibit similar complex behavior — e.g., the human heartbeat dynamics generates fluctuations which are anti-correlated on a wide range of time scales [9] and are also characterized by a broad MF spectrum [10]. Furthermore, it was recently reported that the magnitudes of these fluctuations are positively correlated and exhibit power-law scaling behavior over scales from seconds to hours [11]. These and other examples [12,13] raise the possibility that there may be a relation between the scaling in the magnitude series and the MF spectrum of the original series which is independent of the type of long-range correlations present in the original series.

Here we study the relation between the scaling exponent which quantifies the power-law correlations in the magnitude series and the MF spectrum of artificially generated time series with analytically built-in MF properties. We find that the scaling exponent of the magnitude series is a monotonically increasing function of the MF spectrum width of the original series.

Long-range power-law correlated series are *monofractal* when they are characterized by exponents  $\tau(q)$  for different moments  $q$  being *linearly* depend on  $q$ ,

$$\tau(q) = Hq + \tau(0), \quad (1)$$

with a single Hurst exponent [14],

$$H \equiv d\tau/dq = \text{const.} \quad (2)$$

Other long-range correlated series are *multifractal* and are described by  $\tau(q)$  that depend *nonlinearly* on  $q$ ,

$$h(q) \equiv d\tau/dq \neq \text{const}, \quad (3)$$

where  $h(q)$  is the local Hurst exponent [6]. The MF spectrum of the series is defined by

$$D(h) = h(q)q - \tau(q), \quad (4)$$

where for monofractal series it collapses to a single point,

$$D(H) = -\tau(0). \quad (5)$$

Here we show that the magnitude series scaling exponent is related to the width of the MF spectrum  $D(h)$  of the original series,

$$\Delta h \equiv h_{\max} - h_{\min}, \quad (6)$$

where  $h_{\min}$  and  $h_{\max}$  are the minimum and maximum of the local Hurst exponent.

We use the algorithm proposed in [15] to generate artificial MF series. Briefly, the random MF series is built by specifying discrete wavelet coefficients which are constructed recursively using a random variable  $W$  (see [15,16] for details). Once the wavelet coefficients are obtained, we apply inverse wavelet transform to generate the MF random series; we use the 10-tap Daubechies discrete wavelet transform [16,17]. The MF properties of the obtained artificial series depend on the form of distribution from which the random variable  $W$  is chosen.

We consider two different types of probability distributions for the random variable  $W$ , the log-normal distribution and the log-Poisson distribution. For these two examples the MF properties are known analytically [15]. The MF spectrum  $D(h)$  is symmetric for the log-normal distribution and asymmetric for the log-Poisson distribution.

First we consider the case of a log-normal random variable  $W$  (Fig. 1a) — i.e.,  $\ln|W|$  is normally distributed and  $\mu$ ,  $\sigma^2$  are the mean and the variance of  $\ln|W|$ . We choose  $\mu = -(\ln 2)/4$ . In this case the scaling exponents,  $\tau(q)$ , are given by [15]

$$\tau(q) = -\frac{\sigma^2}{2 \ln 2} q^2 + \frac{q}{4} - 1, \quad (7)$$

and the MF spectrum  $D(h)$  is

$$D(h) = -\frac{(h - 1/4)^2 \ln 2}{2\sigma^2} + 1. \quad (8)$$

We calculate the MF spectrum width by solving  $D(h) = 0$  and find

$$\Delta h = \left( \frac{\sqrt{2}\sigma}{\sqrt{\ln 2}} + \frac{1}{4} \right) - \left( -\frac{\sqrt{2}\sigma}{\sqrt{\ln 2}} + \frac{1}{4} \right) = \frac{2\sqrt{2}\sigma}{\sqrt{\ln 2}}. \quad (9)$$

For  $\sigma = 0$ , the series is monofractal, while for large value of  $\sigma$  it has a broad MF spectrum. The MF spectrum is symmetric around  $D(1/4) = 1$  and collapses to a single point  $D(1/4) = 1$  when  $\sigma \rightarrow 0$  [Eq. (8) and Fig. 2a,b].

We generate series with different  $\sigma$  values ranging from 0 to 0.1 (20 realizations for each  $\sigma$ ) and calculate the magnitude scaling exponent using the method of detrended fluctuation analysis (DFA) [18] (see Fig. 1c and Fig. 3). We find that the scaling exponent of the magnitude series increases monotonically with  $\sigma$  from uncorrelated magnitude series with scaling exponent  $\approx 0.5$  for  $\sigma = 0$  to strongly correlated magnitude series with exponent  $\approx 1$  for  $\sigma = 0.1$ . Further, the magnitude exponent converges to 1.

Next we show that the scaling exponent of the magnitude does *not* only depend on the positive moments (and especially the fourth moment) but rather relates to the entire MF spectrum. For this purpose we study an example with an asymmetric MF spectrum where the exponents for negative moments,  $h(q < 0)$ , are changed drastically when the MF spectrum width is changed, while the exponents for the positive moments  $h(q > 0)$  are less significantly changed. We tune the parameters in such a way that the fourth moment  $\tau(4)$  is fixed and the second moment  $\tau(2)$  is hardly changed.

To generate a series with an asymmetric MF spectrum we consider the case of a random variable  $W$  from a log-Poisson distribution, where  $\lambda$  is the mean and the variance of a Poisson distributed variable  $P$ , and  $\ln|W|$  has the same distribution function as  $P \ln \delta + \gamma$ . We choose  $\lambda = \ln 2$  and  $\gamma = -\delta^4 (\ln 2)/4$  where  $\delta$  is an appropriately chosen positive parameter [15]. Under this choice of parameters,

$$\tau(q) = -\delta^q + q\delta^4/4, \quad (10)$$

and

$$D(h) = \frac{1}{\ln \delta} \left( h - \frac{\delta^4}{4} \right) \left[ \ln \left( \frac{h - \delta^4/4}{-\ln \delta} \right) - 1 \right]. \quad (11)$$

Thus the fourth moment  $\tau(4)$  is independent of  $\delta$

$$\tau(4) = 0, \quad (12)$$

and the width of the MF spectrum depends on  $\delta$

$$\Delta h = (\delta^4/4 - e \ln \delta) - (\delta^4/4) = -e \ln \delta. \quad (13)$$

For  $\delta = 1$  the MF spectrum collapses to a single point (monofractal) with Hurst exponent  $H = 1/4$  and  $D(1/4) = 1$ . For both cases, decreasing or increasing  $\delta$ , the MF spectrum becomes broader. Here we change the MF spectrum width by decreasing  $\delta$ . The major change in the MF spectrum occurs for negative moments  $q < 0$  (Fig. 2 c, and d). We find that the scaling exponent of the magnitude series increases monotonically with the MF spectrum width (Fig. 3) although the positive moments are hardly changed. Thus, the magnitude series scaling exponent is related to the entire MF spectrum of the original series and not just to the positive moments.

We summarize the results of the log-normal and the log-Poisson examples in Fig. 3 — the magnitude scaling exponent is plotted versus the width of the MF spectrum [Eqs. (9),(13)]. Surprisingly, these two examples collapse on the same curve. This collapse suggests a possible one-to-one relation between the magnitude scaling exponent and the MF spectrum width [19].

The relation between the scaling exponent of the magnitude series and the MF spectrum width of the original series is consistent with recent works on MF random walk models with built-in correlations [20]. Recently [11] it was found that power-law correlations in the magnitude series are related to nonlinear [21] properties of the increment series  $\Delta x_i$  — nonlinear series have correlated magnitude series while linear series have uncorrelated magnitude series. On the other hand, monofractal series are linear series while MF series series are nonlinear [10]. Thus the origin of the magnitude series correlations and the corresponding MF spectrum may be related through the nonlinearity of the original time series.

The results described here may be important from a practical point of view. The calculation of the MF spectrum from a time series involves advanced numerical techniques [6] and requires long time series. Our analysis of the magnitude series is less sophisticated than the MF analysis and is applicable to shorter time series than one need for the MF analysis. We note that a direct numerical calculation of the MF spectrum width from a time series may not follow the monotonic relation described in Fig. 3 due to overestimation of the MF spectrum width caused by (i) the numerical technique for calculating the MF spectrum [and estimation of  $\tau(q \rightarrow \pm\infty)$ ] and by (ii) finite series length. However, we expect the magnitude scaling exponent to increase monotonically with the width of the MF spectrum [12,19] since both are related to the nonlinearity of the original series.

Partial support was provided by the NIH/National Center for Research Resources (P41 RR13622), the

Israel-USA Binational Science Foundation, and the German Academic Exchange Service (DAAD). We thank A.L. Goldberger and J.W. Kantelhardt for helpful discussions.

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- [19] We also performed the same procedure (as in Figs. 1,3) for a deterministic multifractal object, namely, the generalized devil's staircase [6]. Also here we find that magnitude exponent increases monotonically with the multifractal spectrum width. However, for very small multifractal width the magnitude exponent was  $\sim 0$  and then increased sharply to the behavior shown in Fig. 3; this behavior might be related to the deterministic nature of this multifractal object. We also note that counter examples for which broad MF spectrum is not associated with magnitude series correlations (or vice versa) may exist.
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ment) from the power spectrum and the probability distribution alone, regardless of the Fourier phases [22]. This definition includes autoregression processes ( $x_n = \sum_{i=1}^M a_i x_{n-i} + \sum_{i=0}^L b_i \eta_{n-i}$  where  $\eta$  is Gaussian white noise) and fractional Brownian motion; the output,  $x_n$ , of these processes may undergo monotonic nonlinear transformations  $s_n = s(x_n)$  and still be linear. Processes which do not follow the above rule are defined as *nonlinear*.

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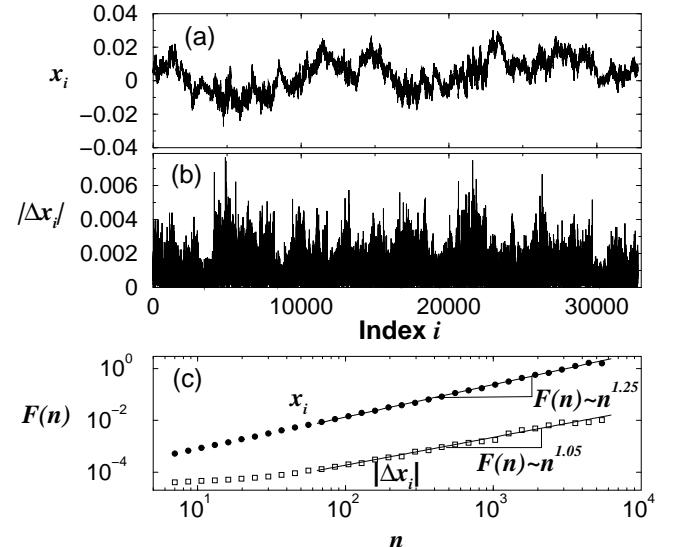


FIG. 1. (a) An example of 32,768 data points of an artificial MF random series with Hurst exponent of  $H = 0.25$ . We use the log-normal  $W$  wavelet cascade with  $\mu = -(\ln 2)/4$  and  $\sigma = 0.1$  to generate the artificial series. (b) The magnitudes of increments obtained from the original series shown in (a). Patches of more “volatile” increments with large magnitude are followed by patches of less volatile increments with small magnitude. In contrast, a monofractal series in (a) would result in a homogeneous series for the magnitudes of increments  $|\Delta x_i|$ . (c) The root mean square fluctuations  $F(n)$  versus the window size  $n$  (obtained by the second order detrended fluctuation analysis [18]) for the original series shown in (a) and for the magnitude series shown in (b). The scaling exponents are calculated for window scales  $n > 64$ . The correlations of the original series follow a scaling law  $F(n) \sim n^{1.25}$  and is consistent with a theoretical Hurst exponent of  $H = 0.25$  since  $H = \alpha - 1$  (the difference of 1 is due to integration in the DFA procedure). The magnitude series is also correlated with an exponent of  $\alpha \approx 1$ .

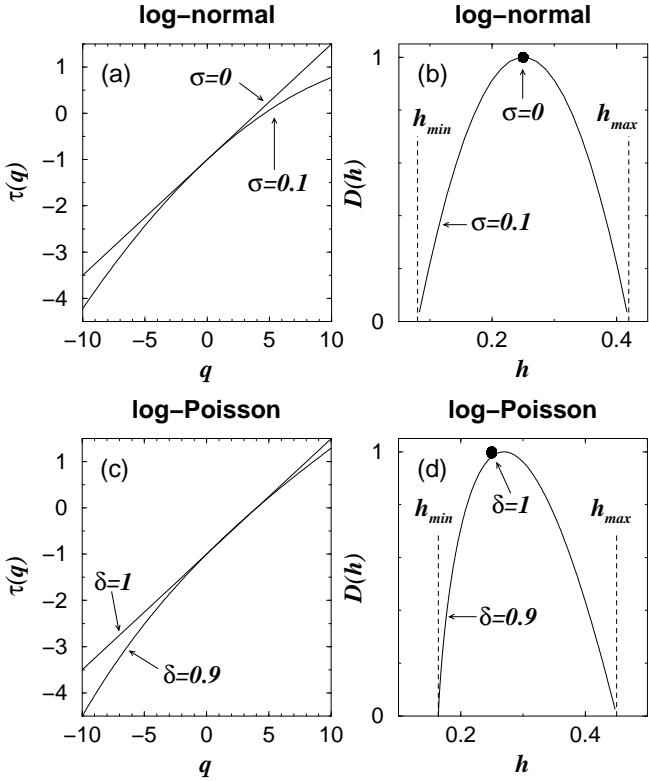


FIG. 2. (a) Theoretical scaling exponent  $\tau(q)$  for  $\sigma = 0$  and  $\sigma = 0.1$  for a MF series generated using random variable  $W$  from a log-normal distribution. The series has symmetric MF spectra where the width of the MF spectrum is proportional to  $\sigma$  [Eq. (9)]. For  $\sigma = 0$   $\tau(q)$  is linear and the series is monofractal while for  $\sigma = 0.1$   $\tau(q)$  is curved and the series is MF. (b) The MF spectrum,  $D(h)$ , obtained from  $\tau(q)$  shown in (a). For  $\sigma = 0$  ( $\bullet$ ) the  $D(h)$  MF spectrum is a single point indicating monofractal behavior with a global Hurst exponent  $H = 0.25$ . For  $\sigma = 0.1$  (curved solid line) the  $D(h)$  spectrum is wide indicating MF behavior. The MF spectrum is symmetric and centered around  $h = 0.25$ . The vertical dashed lines indicate the minimum ( $h_{min}$ ) and maximum ( $h_{max}$ ) values of the local Hurst exponent. (c) Same as (a) for a MF series generated using random variable  $W$  from a log-Poisson distribution. In this case the MF spectrum is *asymmetric* for which the second moment  $\tau(2)$  and the forth moment  $\tau(4)$  are almost fixed. The  $\tau(q)$  spectrum is given for  $\delta = 1$  (linear dependence that indicates monofractality) and for  $\delta = 0.9$  (curved line that indicates multifractality). Note that  $\tau(q)$  remains almost unchanged for positive  $q$ 's while  $\tau(q)$  changes significantly for negative  $q$ 's. (d) The MF spectrum,  $D(h)$ , for the examples shown in (c). The  $D(h)$  spectrum for  $\delta = 1$  is monofractal ( $\bullet$ ); the global Hurst exponent is  $H = 0.25$ . For  $\delta = 0.9$  (solid curved line)  $D(h)$  MF spectrum is broad and asymmetric.  $D(h)$  is more stretched to the right indicating larger changes in the negative moments.

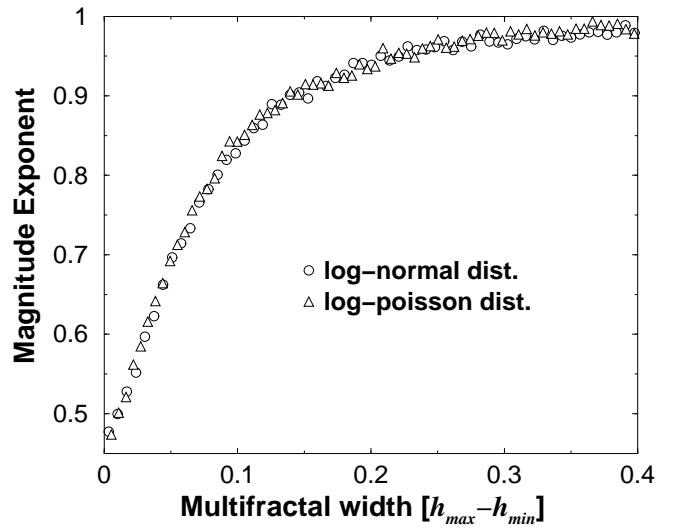


FIG. 3. The magnitude scaling exponent versus the analytical MF spectrum width,  $\Delta h \equiv h_{max} - h_{min}$ . For each point we generate 20 realizations each of 65,536 points. We calculate the magnitude scaling exponent in the range  $64 \leq n \leq 4096$  using the second order DFA; the average exponent is shown (the standard deviation is less than 0.05). Both examples, log-normal distribution ( $\circ$ ) and log-Poisson distribution ( $\triangle$ ), exhibit the same behavior of monotonically increasing magnitude exponent as a function of the MF spectrum width. We approximate this increase by  $1/(1 + e^{-17\Delta h})$ . In contrast, the value of the 2-point scaling exponent of the original series does not depend on the width of the MF spectrum.